RESIDUE CURRENTS ASSOCIATED WITH WEAKLY HOLOMORPHIC FUNCTIONS

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Abstract. We construct Coleff-Herrera products and Bochner-Martinelli type residue currents associated with a tuple f of weakly holomorphic functions, and show that these currents satisfy basic properties from the (strongly) holomorphic case, as the transformation law, the Poincaré-Lelong formula and the equivalence of the Coleff-Herrera product and the Bochner-Martinelli type residue current associated with f when f defines a complete intersection.

1. Introduction

The basic example of a residue current, introduced by Coleff and Herrera in [12], is a current called the *Coleff-Herrera product* associated with a strongly holomorphic mapping $f = (f_1, \dots, f_p)$. The Coleff-Herrera product is defined by

$$\bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} . \varphi = \lim_{\delta \to 0^+} \int_{\cap \{|f_i| = \epsilon_i\}} \frac{\varphi}{f_1 \dots f_p}, \tag{1.1}$$

where φ is a test-form and $\epsilon(\delta)$ tends to 0 along a so-called *admissible path*, which means essentially that $\epsilon_1(\delta)$ tends to 0 much faster than $\epsilon_2(\delta)$ and so on, for the precise definition, see [12]. The Coleff-Herrera product was defined over an analytic space, however, most of the work on residue currents thereafter has focused on the case of holomorphic functions on a complex manifold. The theory of residue currents has various applications, for example to effective versions of division problems etc., see for example [3], [9], [22] and the references therein.

On an analytic space Z, the most common notion of holomorphic functions are the *strongly holomorphic* functions, that is, functions which are locally the restriction of holomorphic functions in any local embedding. In some cases, this can be a little too restrictive, and the *weakly holomorphic* functions might be more natural. These are functions defined on Z_{reg}, which are holomorphic on Z_{reg} and locally bounded at Z_{sing}. Two reasons why these are natural: weakly holomorphic functions are the integral closure of the strongly holomorphic functions in the ring of meromorphic functions, and weakly holomorphic functions correspond to strongly holomorphic functions in any

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normal modification of Z. A slightly better behaved but more restrictive notion are the *c-holomorphic* functions, functions which are weakly holomorphic and continuous on all of Z.

In a recent article [13], Denkowski introduced a residue calculus for c-holomorphic functions, and showed that this calculus satisfies many of the basic properties known from the strongly holomorphic or smooth cases. It is then a natural question to ask what happens in the case of weakly holomorphic functions. However, as in the c-holomorphic case, it is not obvious how to define the associated residue currents.

In the strongly holomorphic case, there are various ways to define the Coleff-Herrera product (for the equivalence of various definitions of the Coleff-Herrera product, also in the non complete intersection case, see for example [18]). The definition we will use is based on analytic continuation as in [24], which was inspired by the ideas in [6] and [7] that the principal value current $1/f$ of a holomorphic function f can be defined by $(|f|^{2\lambda}/f)|_{\lambda=0}$. If $f = (f_1, \ldots, f_p)$ is strongly holomorphic on Z, we define the Coleff-Herrera product of f by

$$\left.\frac{\bar\partial|f_p|^{2\lambda_p} \wedge \cdots \wedge \bar\partial|f_1|^{2\lambda_1}}{f_1 \cdots f_p}\right|_{\lambda_1=0,\ldots,\lambda_p=0}$$

(where we by $|_{\lambda_1=0,\ldots,\lambda_p=0}$ mean that we take the analytic continuation in λ_1 to $\lambda_1 = 0$, then in λ_2 and so on, see Section 4 for details). Recall that a *modification* of an analytic space Z is a proper surjective holomorphic mapping $\pi : Y \to Z$ from an analytic space Y such that there exists a nowhere dense analytic set $E \subset X$ with $\pi|_{Y\setminus\pi^{-1}(E)} : Y \setminus \pi^{-1}(E) \to X \setminus E$ a biholomorphism. It is easy to see by analytic continuation, that if $\pi : Y \to Z$ is a modification of Z, then the Coleff-Herrera product of f can be defined as the push-forward of the Coleff-Herrera product of $f' := \pi^* f$. For weakly holomorphic functions, we can use this observation to define the Coleff-Herrera product, since the pull-back of a weakly holomorphic function to the normalization is strongly holomorphic. If f is weakly holomorphic, we define the Coleff-Herrera product of f by

$$\mu^f := \bar\partial\frac{1}{f_p} \wedge \cdots \wedge \bar\partial\frac{1}{f_1} := \pi_*\left(\bar\partial\frac{1}{f'_p} \wedge \cdots \wedge \bar\partial\frac{1}{f'_1}\right), \tag{1.2}$$

where $f' = \pi^* f$. By the observation above, this of course coincides with the usual definition in case of strongly holomorphic functions, and this definition is also consistent with the one in [13] in the case of c-holomorphic functions, see Proposition 4.1.

Because of our definition, the properties we prove of the Coleff-Herrera product for weakly holomorphic functions can mostly be reduced (by going back to the normalization) to the strongly holomorphic case. Thus the main part of this article concerns giving a coherent

exposition of the basic theory of residue currents in the strongly holomorphic case. This is done based on analytic continuation of currents and the notion of pseudomeromorphic currents as introduced in [4], which is developed with holomorphic functions on a complex manifold. We will see that this approach works well also with strongly holomorphic functions on an analytic space, and we believe that this might be of independent interest, although most of the results should be known.

However, even for the statement of these properties in the weakly holomorphic case, two problems occur, namely how is multiplication of a weakly holomorphic function with a current defined, and what is the zero set of a tuple of weakly holomorphic functions? And hence also, what should a complete intersection mean?

For the problem of multiplication of weakly holomorphic functions with currents, we take a similar approach as for the definition of the Coleff-Herrera product. Namely, if $\pi : Y \to Z$ is a modification, μ is a current on Z, g is strongly holomorphic on Z and $\mu = \pi_* \mu'$, then

$$g\mu = \pi_*(\pi^* g\mu'). \tag{1.3}$$

The right hand side of (1.3) still exist if g is weakly holomorphic on Z and Y is normal, so we take this as a definition of $g\mu$. However, that this is well-defined depend on the fact that we have a certain "canonical" representative of the Coleff-Herrera product in the normalization (or any normal modification). We will see in Section 5 that (1.3) depends on the choice of representative μ' and can thus not be used to define a general multiplication of weakly holomorphic with currents on Z.

For the zero set of one weakly holomorphic function, all reasonable definitions should coincide. For the zero set of a weakly holomorphic mapping f, it is natural to take into account that the zero sets of the indivudual components of f can "belong" to different irreducible components. We introduce in Section 2 a notion of common zero set of f, depending on f as a mapping, and not only on the individual components, which however may differ from the intersection of the respective zero sets.

The Coleff-Herrera product μ^f in (1.2) associated with a strongly holomorphic mapping $f = (f_1, \dots, f_p)$ satisfies

$$\operatorname{supp} \mu^f \subseteq Z_f \quad \text{and} \quad \bar\partial \mu^f = 0.$$

In addition, if f forms a complete intersection, the Coleff-Herrera product is alternating in the residue factors and

$$(f_1, \dots, f_p) \subseteq \operatorname{ann} \mu^f,$$

where $(f_1, \dots, f_p)$ is the ideal generated by $f_1, \dots, f_p$. We also have the *transformation law* for residue currents (see [14]), which says that if $f = (f_1, \dots, f_p)$ and $g = (g_1, \dots, g_p)$ define a complete intersection, and there exists a matrix A of holomorphic functions such that $g = Af$,

then

$$(\det A)\bar\partial\frac{1}{g_p}\wedge\cdots\wedge\bar\partial\frac{1}{g_1}=\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}.$$

The *Poincaré-Lelong formula* relates the Coleff-Herrera product of f and the integration current $[Z_f]$ on Z_f (with multiplicities) and it says that

$$\frac{1}{(2\pi i)^p}\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge df_1\wedge\cdots\wedge df_p=[Z_f].$$

We will see that in fact all those statements still hold also in the weakly holomorphic case. However, as mentioned above, zero sets of weakly holomorphic functions and multiplication of currents with weakly holomorphic functions need to be interpreted in the right way.

Bochner-Martinelli type residue currents were first introduced in [21] by Passare, Tsikh and Yger (on a complex manifold) as an alternative way of defining a residue current corresponding to a tuple of holomorphic functions. In [23], Bochner-Martinelli type residue currents were used to prove a generalizations of Jacobi's residue formula, and in [10] they were also constructed on an analytic space to prove a similar result on an affine algebraic variety.

The Bochner-Martinelli type residue currents give another reason why our definition of Coleff-Herrera product is a natural one. In the smooth case, it was proved in [21] that if the functions define a complete intersection, then the Coleff-Herrera product and the Bochner-Martinelli current coincide. It is suggested in [10] that the same statement holds in the singular case with a similar proof. We will construct Bochner-Martinelli type residue currents associated with a tuple of weakly holomorphic functions, and we will show that this equality holds both in the strongly and weakly holomorphic cases. Another advantage of the Bochner-Martinelli current in the weakly holomorphic case is that it can be defined intrinsically on Z as the analytic continuation of a arbitrarily smooth (depending on a parameter λ) form on Z, see Remark 4.

We have the inclusion $(f_1,\dots,f_p)\subseteq \operatorname{ann}\mu^f$ if f defines a complete intersection, both in the strongly and weakly holomorphic case. This is one direction of the *duality theorem* proven in [14] and [20], which says that if f defines a complete intersection on a complex manifold, we have locally that $(f_1,\dots,f_p)=\operatorname{ann}\mu^f$. However, one can show that depending on certain singularity subvarieties of Z compared to the zero set Z_f, in many cases the inclusion $(f_1,\dots,f_p)\subseteq\operatorname{ann}\mu^f$ is strict, even on a normal analytic space. Thus, one can not expect the duality theorem to hold unrestrictedly on an analytic space, either in the weakly or strongly holomorphic setting. This is the topic of a forthcoming article, and we will not go in to this further in this article.

2. Zero sets of weakly holomorphic functions

The behavior of the currents we define will depend in a crucial way on the zero sets of the weakly holomorphic functions, and in this section we will define the zero set of a weakly holomorphic mapping.

Definition 1. Let $f \in \tilde{\mathcal{O}}(Z)$, where $\tilde{\mathcal{O}}(Z)$ denotes the ring of weakly holomorphic functions on Z. If f is not identically zero on any irreducible component of Z, we define the *zero set* of f by $Z_f := \{z \in Z \mid (1/f)_z \notin \tilde{\mathcal{O}}_z\}$, and if f vanishes identically on the irreducible components Z_α of Z, then f does not vanish identically on any irreducible component of $Z' := \overline{Z \setminus \cup_\alpha Z_\alpha}$, and we define $Z_f := \cup_\alpha Z_\alpha \cup Z_{f|_{Z'}}$.

For any meromorphic function ϕ, there is a standard notion of zero set of ϕ, that we denote by Z'_ϕ, which is defined by $Z'_\phi = \{z \in Z \mid (1/\phi)_z \notin \mathcal{O}_z\}$. Since weakly holomorphic functions are meromorphic, this gives another definition of zero set if f is a weakly holomorphic function. Clearly $Z_f \subseteq Z'_f$, but note that in general the inclusion can be strict, so the two definitions does not coincide.

Remark 1. We have $z \in Z_f$ if and only if there exists a sequence $z_i \to z$ with $z_i \in Z_{\text{reg}}$ such that $f(z_i) \to 0$ (since if we cannot find such a sequence, then $1/f$ is weakly holomorphic); hence when f is c-holomorphic, Z_f coincides with the zero set when seen as continuous function.

We will use the following characterization of the zero set of a weakly holomorphic function. However, since this is a special case of Proposition 2.3, we omit the proof.

Lemma 2.1. *Let $\pi : Z' \to Z$ be the normalization of Z. If $f \in \tilde{\mathcal{O}}(Z)$, then Z_f is an analytic subset of Z, and $Z_f = \pi(Z_{\pi^* f})$.*

To study the dimension of zero sets of weakly holomorphic functions, we will need the following lemma, which shows that subvarieties of the normalization correspond to subvarieties of Z of the same dimension, and vice versa.

Lemma 2.2. *Let $\pi : Z' \to Z$ be the normalization of Z. If Y' is a subvariety of Z', then $\pi(Y')$ is a subvariety of Z with $\dim Y' = \dim \pi(Y')$, and if Y is a subvariety of Z, then $\pi^{-1}(Y)$ is a subvariety of Z' with $\dim Y = \dim \pi^{-1}(Y)$.*

Proof. The first part follows from Remmert's proper mapping theorem, when formulated as for example in [15], since π is a finite proper holomorphic mapping. Hence we get from the first part that $\dim \pi^{-1}(Y) = \dim \pi(\pi^{-1}(Y)) = \dim Y$, where the second equality holds since π is surjective. □

If $f \in \tilde{\mathcal{O}}(Z)$ and $f \not\equiv 0$ on any irreducible component of Z, then $\operatorname{codim} Z_f = 1$ or $Z_f = \emptyset$. In fact, if $f' = \pi^* f$ and $Z_{f'} \neq \emptyset$, then f'

is strongly holomorphic, and $Z_{f'} = \{f' = 0\}$ has codimension 1, and since $Z_f = \pi(Z_{f'})$ by Lemma 2.1, Z_f has codimension 1 by Lemma 2.2. However, even though the zero set of a weakly holomorphic function has codimension 1, as we will see in the next example, the zero set is in general not the zero set of one single strongly holomorphic function, even for c-holomorphic functions on an irreducible space.

Example 1. Let $V = \{z_1^3 - z_2^2 = z_3^3 - z_4^2 = 0\} \subset \mathbb{C}^4$. Then V has normalization $\pi : \mathbb{C}^2 \to V$, $\pi(t_1, t_2) = (t_1^2, t_1^3, t_2^2, t_2^3)$, and hence $f = z_2/z_1 - z_4/z_3$ is c-holomorphic since $\pi^* f = t_1 - t_2$. The set $Z_f = \{(t^2, t^3, t^2, t^3)\}$ has codimension 1 in Z, but there does not exist a holomorphic function in a neighborhood of 0 such that $f(t_1^2, t_1^3, t_2^2, t_2^3) = 0$ exactly when $t_1 = t_2$, since in that case, we could write $f(t_1^2, t_1^3, t_2^2, t_2^3) = (t_1 - t_2)^m u(t_1, t_2)$ for some $m \in \mathbb{N}$, where $u(0,0) \neq 0$, which is easily seen to be impossible. Hence, Z_f is not the zero set of one single strongly holomorphic function.

Example 2. Let $Z = Z_1 \cup Z_2 \subset \mathbb{C}^6$, where $Z_1 = \mathbb{C}^3 \times \{0\}$ and $Z_2 = \{0\} \times \mathbb{C}^3$. Define the functions f and g by

$$f(z) = \begin{cases} z_1 & z \in Z_1 \setminus \{0\} \\ 1 & z \in Z_2 \setminus \{0\} \end{cases} \text{ and } g(z) = \begin{cases} 1 & z \in Z_1 \setminus \{0\} \\ z_4 & z \in Z_2 \setminus \{0\} \end{cases}.$$

Then $f, g \in \tilde{\mathcal{O}}(Z)$, and $Z_f = Z_1 \cap \{z_1 = 0\}$, and $Z_g = Z_2 \cap \{z_4 = 0\}$ which both have codimension 1 in Z. However, $Z_f \cap Z_g = \{0\}$, which has codimension 3. Hence, zero sets of weakly holomorphic functions do not behave as well as one could hope with respect to intersections. If we let $f_1 = f_2 = f$, $f_3 = g$, then $Z_{f_1} \cap Z_{f_2} \cap Z_{f_3} = \{0\}$ has codimension 3, while $Z_{f_1} \cap Z_{f_2} = Z_f$ has codimension 1 at 0 in Z. Hence, if one defines a complete intersection for zero sets of weakly holomorphic functions $f = (f_1, \cdots, f_p)$ by requiring that $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension p in Z, then it will not follow in general that $(Z_{f_1} \cap \cdots \cap Z_{f_k}, z)$ has codimension k for $z \in Z_{f_1} \cap \cdots \cap Z_{f_p}$.

Remark 2. Note that for c-holomorphic functions $f = (f_1, \cdots, f_p)$, if $f' = \pi^* f$, where $\pi : Z' \to Z$ is the normalization, then $\pi(Z_{f_1'} \cap \cdots \cap Z_{f_p'}) = Z_{f_1} \cap \cdots \cap Z_{f_p}$. Thus if we say that $f = (f_1, \cdots, f_p)$, where $f_i \in \mathcal{O}_c(Z)$, forms a complete intersection in Z if $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension p, then this holds if and only if f' forms a complete intersection in Z' by Lemma 2.2.

As we see in Example 2, this does not hold for weakly holomorphic functions, because there, $Z_f \cap Z_g = \{0\}$, while $Z_{f'} \cap Z_{g'} = \emptyset$. Thus, the straight forward generalization of complete intersection, where the zero set $Z_{f_1} \cap \cdots \cap Z_{f_p}$ is required to have codimension p does not share the same good properties in the weakly holomorphic case as in the strongly holomorphic (or c-holomorphic) case. Because of this, we will use a different definition of both the common zero set of weakly holomorphic

functions and of a complete intersection. However, it coincides with the usual definitions in case of strongly holomorphic or c-holomorphic functions, and with our definition the problems above disappear.

Definition 2. Let $f = (f_1, \cdots, f_p)$ be weakly holomorphic. We define the *common zero set* of f, denoted by Z_f, as the set of $z \in Z$ such that there exists a sequence $z_i \in Z_{\text{reg}}$ with $z_i \to z$, and $f_k(z_i) \to 0$ for $k = 1, \cdots, p$. We will see that Z_f is an analytic subset of Z, and hence we can say that f forms a *complete intersection* if Z_f has codimension p in Z.

Note that by Remark 1, this definition is consistent with the definition of Z_f in the case of one function. We also see that in Example 2, $Z_{(f,g)} = \emptyset$, and hence is not a complete intersection in our sense. Just as for one function, we can give a characterization of the zero set with the help of the normalization.

Proposition 2.3. *Let $f = (f_1, \cdots, f_p)$ be weakly holomorphic, and let $f' = \pi^* f$, where $\pi : Z' \to Z$ is the normalization. Then*

$$Z_f = \pi(Z_{f'_1} \cap \cdots \cap Z_{f'_p}), \tag{2.1}$$

and Z_f is an analytic subset of Z of codimension $\leq p$. In general,

$$Z_f \subseteq Z_{f_1} \cap \cdots \cap Z_{f_p}, \tag{2.2}$$

with equality if f is c-holomorphic. In addition, f is a complete intersection if and only if f' is a complete intersection in the normalization.

Proof. If $z' \in Z_{f'_1} \cap \cdots \cap Z_{f'_p}$, then we can take a sequence $z'_i \to z'$ such that $z'_i \in \pi^{-1}(Z_{\text{reg}})$. Then, if we let $z_i = \pi(z'_i)$, we get that $f_k(z_i) \to 0$, and hence we have the inclusion $Z_f \supseteq \pi(Z_{f'_1} \cap \cdots \cap Z_{f'_p})$ in (2.1). For the other inclusion, if we have a sequence $z_i \to z$ such that $z \in Z_f$, since π is proper we can choose a convergent subsequence $z'_{k_i} \to z'$ such that $\pi(z'_{k_i}) = z_{k_i}$, and since $z \in Z_f$, we must have $f'(z') = 0$, so $\pi(z') = z$, with $z' \in Z_{f'_1} \cap \cdots \cap Z_{f'_p}$. Now, the fact that Z_f is an analytic subset of Z follows by (2.1) and Remmert's proper mapping theorem, since $Z_{f'_i}$ are analytic subsets of Z'. Since f' is strongly holomorphic, $Z_{f'}$ has codimension $\leq p$, so by (2.1) combined with Lemma 2.2 we get that Z_f has codimension $\leq p$. If f is c-holomorphic, the equality in (2.2) follows by (2.1) since for any continuous mapping f, $Z_{f_1} \cap \cdots \cap Z_{f_p} = \pi(Z_{\pi^* f_1} \cap \cdots \cap Z_{\pi^* f_p})$, and the general case also follows from (2.1) since $\pi(Z_{f'_1} \cap \cdots \cap Z_{f'_p}) \subseteq \pi(Z_{f'_1}) \cap \cdots \cap \pi(Z_{f'_p}) = Z_{f_1} \cap \cdots \cap Z_{f_p}$. Finally, the fact that f is a complete intersection if and only if f' is a complete intersection follows from (2.1) together with Lemma 2.2. $\square$

We note that if $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension $\geq p$, then either $Z_f = \emptyset$, or Z_f has codimension p since by Proposition 2.3, $Z_f \subseteq Z_{f_1} \cap \cdots \cap Z_{f_p}$, and Z_f has codimension at most p. Thus, if $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension $\geq p$, and some result depends on the fact that Z_f should

have codimension $\geq p$, it will still be true with this other definition of complete intersection, and this will be the case for all results about residue currents stated here, except for the Poincaré-Lelong formula, Proposition 8.1. Hence, our definition of complete intersection is not essential for the results to hold, however, the other definition will in general give weaker statements, since it might very well happen that $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension $< p$, while Z_f has codimension p.

Note also that, if $f = (f_1, \cdots, f_p)$ is a complete intersection and $f_0 = (f_1, \cdots, f_k)$, then (Z_{f_0}, z) has codimension k for $z \in Z_f$, since if $z' \in \pi^{-1}(z)$, then $(Z_{f'_0}, z')$ has codimension k, and hence since π is a finite proper holomorphic mapping, $(Z_{f_0}, z) = \cup_{z'_j \in \pi^{-1}(z)} \pi((Z_{f'_0}, z'_j))$ has codimension k in Z.

3. Pseudomeromorphic currents on an analytic space

We will in this section introduce pseudomeromorphic currents on an analytic space. Pseudomeromorphic currents on a complex manifold were introduced by Andersson and Wulcan in [4], and it was observed that currents like the Coleff-Herrera product and Bochner-Martinelli type residue currents are pseudomeromorphic. Two important properties of pseudomeromorphic currents in the smooth case are the direct analogues of Proposition 3.1 and Proposition 3.2. Since these hold also in the singular case, many properties of residue currents hold also for strongly holomorphic functions by more or less the same argument as in the smooth case.

The pseudomeromorphic currents are intrinsic objects of the analytic space Z, so we begin with explaining what we mean by a current on an analytic space. We will follow the definitions used in [8] and [16]. To begin with, we assume that Z is an analytic subvariety of Ω, for some open set $\Omega \subseteq \mathbb{C}^n$. Then, we define the set of smooth forms of bidegree (p,q) in Z by $\mathcal{E}_{p,q}(Z) = \mathcal{E}_{p,q}(\Omega)/\mathcal{N}_{p,q,Z}(\Omega)$, where $\mathcal{E}_{p,q}(\Omega)$ are the smooth (p,q)-forms in Ω and $\mathcal{N}_{p,q,Z}(\Omega) \subset \mathcal{E}_{p,q}(\Omega)$ are the smooth forms φ such that $i^*\varphi \equiv 0$, where $i : Z_{\text{reg}} \to \Omega$ is the inclusion map. The set of test forms on Z, $\mathcal{D}_{p,q}(Z)$, are the forms in $\mathcal{E}_{p,q}(Z)$ with compact support. With the usual topology on $\mathcal{D}_{p,q}(\Omega)$ by uniform convergence of coefficients of differential forms together with their derivatives on compact sets, we give $\mathcal{D}_{p,q}(Z)$ the quotient topology from the projection $\mathcal{D}_{p,q}(\Omega) \to \mathcal{D}_{p,q}(Z)$. Then, (p,q)-currents on Z, denoted $\mathcal{D}'_{p,q}$, are the continuous linear functionals on $\mathcal{D}_{k-p,k-q}(Z)$, where $k = \dim Z$. However, more concretely, this just means that μ is a (p,q)-current on Z if and only if $i_*\mu$ is a $(n-k+p, n-k+q)$-current in the usual sense on Ω, such that μ vanishes on forms in $\mathcal{N}_{k-p,k-q,Z}(\Omega)$.

It is easy to see that the definitions of smooth forms, test forms and currents are independent of the embedding, and hence by gluing together in the same way one does on a complex manifold, we can define the sheafs of smooth forms, test forms and currents on any analytic

space Z. Note in particular that by a smooth function on Z, we mean a function which is locally the restriction of a smooth function in the ambient space.

In $\mathbb{C}$, one can define the principal value current $1/z^n = |z|^{2\lambda}/z^n|_{\lambda=0}$ by analytic continuation, where $|_{\lambda=0}$ denotes that for $\operatorname{Re}\lambda \gg 0$, we take the action of $|z|^{2\lambda}/z^n$ on a test-form and take the value of the analytic continuation to $\lambda = 0$, which is easily seen to exist by a Taylor expansion, or integration by parts. Thus, if α is a smooth form on $\mathbb{C}^n$ and $\{i_1, \cdots, i_m\} \subseteq \{1, \cdots, n\}$, with i_j disjoint, then one gets a well-defined current

$$\frac{1}{z_{i_1}^{n_1}} \cdots \frac{1}{z_{i_k}^{n_k}} \bar\partial \frac{1}{z_{i_k+1}^{n_{k+1}}} \wedge \cdots \wedge \bar\partial \frac{1}{z_{i_m}^{n_m}} \wedge \alpha \tag{3.1}$$

on $\mathbb{C}^n$ by taking $\bar\partial$ in the current sense together with tensor product of currents and multiplication of currents with smooth forms. In [4], if α has compact support, a current of the form (3.1) is called an *elementary current*. The class of pseudomeromorphic currents on a complex manifold was then introduced as currents that can be written as a locally finite sum of push-forwards of elementary currents. We will use the same definition on an analytic space Z.

Definition 3. Let $\pi_\alpha : Z_\alpha \to Z$ be a family of modifications of Z, where Z_α are complex manifolds. The class of *pseudomeromorphic currents*, denoted $\mathcal{PM}(Z)$ are the currents μ on Z that can be written as a locally finite sum

$$\mu = \sum (\pi_\alpha)_* \tau_\alpha,$$

where τ_α are elementary currents on Z_α.

Note in particular that, if $\pi : \tilde Z \to Z$ is a resolution of singularities of Z, and if $\mu \in \mathcal{PM}(\tilde Z)$, then $\pi_*\mu \in \mathcal{PM}(Z)$. All the currents introduced here are pseudomeromorphic, as we will see directly from the proofs that the currents exist. In [4], it is shown that if f is holomorphic on a complex manifold X, and $T \in \mathcal{PM}(X)$, one can define a multiplication $(1/f)T$ and $\bar\partial(1/f) \wedge T$. The same idea works equally well for strongly holomorphic functions on an analytic space.

Proposition 3.1. *Let f be strongly holomorphic on Z and $T \in \mathcal{PM}(Z)$. Then the currents*

$$\frac{1}{f} T := \left. \frac{|f|^{2\lambda}}{f} T \right|_{\lambda=0} \quad \text{and} \quad \bar\partial \frac{1}{f} \wedge T := \left. \frac{\bar\partial |f|^{2\lambda}}{f} \wedge T \right|_{\lambda=0},$$

where the right hand sides are defined originally for $\operatorname{Re}\lambda \gg 0$, have current-valued analytic continuations to $\operatorname{Re}\lambda > -\epsilon$ for some $\epsilon > 0$, and the values at $\lambda = 0$ are pseudomeromorphic. The current satisfies the Leibniz rule

$$\bar\partial \left(\frac{1}{f} T \right) = \bar\partial \frac{1}{f} \wedge T + \frac{1}{f} \bar\partial T,$$

and $\operatorname{supp}(\bar\partial(1/f)\wedge T)\subseteq Z_f\cap\operatorname{supp}T$. *If $f\neq 0$, then $(1/f)T$ defined in this way coincides with the usual multiplication of T with the smooth function $1/f$.*

Proof. If Z is smooth, this is Proposition 2.1 in [4], except for the last statement. However, if $f\neq 0$, then $|f(z)|^{2\lambda}/f(z)$ is smooth in both λ and z, and analytic in λ, so if ξ is a test form, $T.((|f|^{2\lambda}/f)\xi)$ is analytic in λ, and hence the analytic continuation to $\lambda=0$ coincides with the value $T.((1/f)\xi)$ at $\lambda=0$. The proof in the general case goes through word for word as in the smooth case in Proposition 2.1 in [4]. $\square$

The following property for pseudomeromorphic currents on an analytic space will turn out to be very useful. The crucial point in the proof of the following proposition is that for any analytic subset $W\subseteq Z$ and any $T\in\mathcal{PM}(Z)$, there exist natural restrictions

$$\mathbf{1}_{W^c}T:=|h|^{2\lambda}T|_{\lambda=0}\quad\text{and}\quad\mathbf{1}_W T:=T-\mathbf{1}_{W^c}T$$

where h is a tuple of holomorphic functions such that $W=\{h=0\}$. The restrictions are independent of the choice of such h, and are such that $\operatorname{supp}\mathbf{1}_W T\subseteq W$. This is Proposition 2.2 in [4], and the proof will go through in exactly the same way when Z is an analytic space.

Proposition 3.2. *Assume that $\mu\in\mathcal{PM}(Z)$, and that μ has support on a variety V. If I_V is the ideal of holomorphic functions vanishing on V, then $\bar I_V\mu=0$. If μ is of bidegree $(*,p)$, and V has codimension $\geq p+1$ in Z, then $\mu=0$.*

The proof in the case Z is a complex manifold, Proposition 2.3 and Corollary 2.4 in [4], will go through in the same way also when Z is an analytic space. The final step in the proof that $\mu=0$ in the smooth case is to prove that $\mu=0$ on $V_{\rm reg}$, which is proved with the help of the previous part and by degree reasons, and then by induction over the dimension of V, $\mu=0$. In the singular case, this is done in the same way. Since this is a local statement, we can assume that $Z\subseteq\Omega\subseteq\mathbb{C}^n$, and consider V as a subvariety of Ω. Then, for the same reasons as in the smooth case, we get that $i_*\mu=0$ on $V_{\rm reg}$, and by induction over the dimension of V that $i_*\mu=0$, and hence $\mu=0$.

4. Coleff-Herrera products of weakly holomorphic functions

Let $f_1,\cdots,f_m\in\tilde{\mathcal{O}}(Z)$. We want to define the Coleff-Herrera product

$$T=\frac{1}{f_m}\cdots\frac{1}{f_{p+1}}\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}.$$

If f is strongly holomorphic, one way to define it is by

$$T=\frac{|f_m|^{2\lambda_m}\cdots|f_{p+1}|^{2\lambda_{p+1}}}{f_m\cdots f_{p+1}}\frac{\bar\partial|f_p|^{2\lambda_p}\wedge\cdots\wedge\bar\partial|f_1|^{2\lambda_1}}{f_p\cdots f_1}\bigg|_{\lambda_1=0,\dots,\lambda_m=0},\tag{4.1}$$

which a priori is defined only when $\operatorname{Re}\lambda_i \gg 0$; however, by Proposition 3.1 it has an analytic continuation in λ_1 to $\operatorname{Re}\lambda_1 > -\epsilon$ for some $\epsilon > 0$, and the value at $\lambda_1 = 0$ is pseudomeromorphic. Again, by Proposition 3.1, it has an analytic continuation in λ_2 to $\lambda_2 = 0$ and so on, and hence the value at $\lambda_1 = 0, \cdots, \lambda_m = 0$ exists. Note that if $\pi : Y \to Z$ is any modification of Z, we can define the corresponding Coleff-Herrera product of $f' = \pi^* f$ in Y, and then take the push-forward to Z, and this will give the same current by analytic continuation.

Now, if f is weakly holomorphic, let $\pi : Z' \to Z$ be the normalization of Z, and $f' = \pi^* f$ which is strongly holomorphic on Z'. Hence, the current

$$T' = \frac{1}{f'_m} \cdots \frac{1}{f'_{p+1}} \bar\partial \frac{1}{f'_p} \wedge \cdots \wedge \bar\partial \frac{1}{f'_1} \tag{4.2}$$

exists.

Definition 4. If $f = (f_1, \ldots, f_m)$ is weakly holomorphic, we define the *Coleff-Herrera product* of f by

$$T = \frac{1}{f_m} \cdots \frac{1}{f_{p+1}} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} = \pi_* T', \tag{4.3}$$

where T' is defined by (4.2).

If f is strongly holomorphic, this definition will be the same as the definition in (4.1) since by the remark above, T can be defined as the push-forward from any modification. In addition, if f is weakly holomorphic, it can be defined by the push-forward of the corresponding current in any normal modification, since any normal modification factors through the normalization.

We will call the factors $1/f_i$ the *principal value factors*, and $\bar\partial(1/f_i)$ the *residue factors*. Note that even though here, we let $\lambda_i = 0$ first for the residue factors and then for the principal value factors, it is in no way essential, and the definition makes sense also if the residue and principal value factors are mixed. However, changing the order will in general give a different current, but as we will see, if f_i define a complete intersection, the current will not depend on the order.

Remark 3. The Coleff-Herrera product for $f = (f_1, \ldots, f_p)$ strongly holomorphic is originally defined in [12] as the limit of integrals over $\cap\{f_i = \epsilon_i\}$ as $\epsilon \to 0$, where $\epsilon(\delta)$ tends to 0 along an admissible path, cf. (1.1). When $\epsilon(\delta)$ tends to 0 along an admissible path, this will correspond to taking the analytic continuation to $\lambda = 0$ in the order above, and in fact, for arbitrary f, the definition in (1.1) is equal to the the one in (4.3) defined by analytic continuation, see [18].

In [13] Denkowski gave a definition of the Coleff-Herrera product of f, for f c-holomorphic, and we will see below that his definition coincides with ours in that case. The idea in [13] was to consider the graph of

f, $\Gamma_f = \{(z, f(z)) \in Z \times \mathbb{C}^p_w | z \in Z\}$, and even though f is only c-holomorphic, the graph will be analytic. If $(z, w) \in \Gamma_f$, then $w = f(z)$, and hence on the graph $f_i = w_i$ is a strongly holomorphic function. If Π is the projection from the graph to Z, since f is continuous, Π is a homeomorphism and in particular proper. The Coleff-Herrera product of f was then defined by

$$\bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} = \Pi_* \left(\bar\partial \frac{1}{w_p} \wedge \cdots \wedge \bar\partial \frac{1}{w_1} \right), \tag{4.4}$$

and since $f_i = w_i$ on Γ_f, this would seem like a reasonable definition of the Coleff-Herrera product of f. The next proposition shows, as one might hope, that the definition of Denkowski coincides with ours.

Proposition 4.1. *If $f = (f_1, \dots, f_p)$ is c-holomorphic, then the definition of the Coleff-Herrera product of f in* (4.3) *and in* (4.4) *coincide.*

Proof. In [13] the definition used for the Coleff-Herrera product of strongly holomorphic functions was the one from [12]. However, by Remark 3 we can assume that the definition by analytic continuation is used instead. Let $\pi : Z' \to Z$ be the normalization of Z and $f' = \pi^* f$. We have projections $\Pi : \Gamma_f \to Z$ and $\Pi' : \Gamma_{f'} \to Z'$, where $\Gamma_f \subseteq Z \times \mathbb{C}^p_w$ and $\Gamma_{f'} \subseteq Z' \times \mathbb{C}^p_{w'}$ are the graphs of f and f'. Thus we have a commutative diagram

$$\begin{array}{ccc} \Gamma_{f'} & \xrightarrow{(\pi \times \mathrm{Id})|_{\Gamma_{f'}}} & \Gamma_f \\ \downarrow{\scriptstyle \Pi'} & & \downarrow{\scriptstyle \Pi} \\ Z' & \xrightarrow{\pi} & Z. \end{array} \tag{4.5}$$

We will denote the current $\bar\partial(1/f'_p) \wedge \cdots \wedge \bar\partial(1/f'_1)$ on Z' by $\mu^{f'}$, and similarly for μ^w and $\mu^{w'}$ defined on Γ_f and $\Gamma_{f'}$ respectively. Then $\bar\partial(1/f_p) \wedge \cdots \wedge \bar\partial(1/f_1)$ is defined in (4.3) by $\pi_* \mu^{f'}$, and in (4.4) by $\Pi_* \mu^w$. Now, $(\pi \times \mathrm{Id})|_{\Gamma_{f'}} : \Gamma_{f'} \to \Gamma_f$ is a modification of Γ_f so we have $\mu^w = (\Pi \times \mathrm{Id})_* \mu^{w'}$, and since $\Pi' : \Gamma_{f'} \to Z'$ is a biholomorphism and $w'_i = f'_i$ on $\Gamma_{f'}$ we also have $\mu^{f'} = \Pi'_* \mu^{w'}$. Thus both are the push-forward of the same current in $\Gamma_{f'}$, and since the diagram (4.5) commutes, both will have the same push-forward to Z. $\square$

The next theorems show that the Coleff-Herrera product of weakly holomorphic functions has some properties that are well-known for strongly holomorphic functions on an analytic space (in the case $m = p$ or $m = p + 1$), see [12], or the case of holomorphic functions on a complex manifold, see [19].

Theorem 4.2. *If $f = (f_1, \cdots, f_m)$ is weakly holomorphic, then T, defined by* (4.3)*, satisfies the Leibniz rule*

$$\bar\partial T = \sum_{j=p+1}^{m} (-1)^{m-j} \frac{1}{f_m} \cdots \wedge \bar\partial \frac{1}{f_j} \wedge \cdots \frac{1}{f_{p+1}} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1},$$

and $\operatorname{supp} T \subseteq Z_{(f_1,\cdots,f_p)}$.

Proof. First we assume that f is strongly holomorphic. Then the Leibniz rule follows by analytic continuation, since if $\operatorname{Re}\lambda \gg 0$, we have

$$\bar\partial\left(\frac{|f|^{2\lambda}}{f}\right) = \frac{\bar\partial |f|^{2\lambda}}{f} \text{ and } \bar\partial\left(\frac{\bar\partial |f|^{2\lambda}}{f}\right) = 0,$$

and the weakly holomorphic case follows by taking push-forward from the normalization. For the last part, let T' be the corresponding current in the normalization, and $f' = \pi^* f$ be the pull-back of f to the normalization. Then by Proposition 3.1, $T' = 0$ outside of $Z_{f_i'}$, and hence $\operatorname{supp} T \subseteq \pi(\operatorname{supp} T') \subseteq \pi(Z_{(f_1',\cdots,f_p')}) = Z_{(f_1,\cdots,f_p)}$, where the last equality follows from Proposition 2.3. □

To be able to state the next theorem, we will need to have a definition of multiplication of the current T in (4.3) with a weakly holomorphic function. If $g \in \tilde{\mathcal{O}}(Z)$, we can define multiplication of the current T defined in (4.3) with g by

$$gT = \pi_*(\pi^* g T'), \tag{4.6}$$

where $\pi : Z' \to Z$ is the normalization of Z, and T' is the corresponding Coleff-Herrera product of $f' = \pi^* f$. This is of course well-defined, but we will see later that it seems hard to define a multiplication of a weakly holomorphic function with a more general current on Z. If all the functions are c-holomorphic, by a similar argument as that in Proposition 4.1, one sees that this definition of multiplication with a c-holomorphic function will give the same current as the one defined in [13].

Theorem 4.3. *Let $f = (f_1, \cdots, f_m)$ be weakly holomorphic, such that $(f_1, \ldots, f_p)$ defines a complete intersection, and that $(f_1, \ldots, f_p, f_i)$ defines a complete intersection for $p+1 \le i \le m$. Then the principal value factors in*

$$T = \frac{1}{f_m} \cdots \frac{1}{f_{p+1}} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1}$$

commute with other principal value factors or residue factors, and the residue factors anticommute. In addition, if $p+1 \le k \le m$, we have

$$f_k T = \frac{1}{f_m} \cdots \widehat{\frac{1}{f_k}} \cdots \frac{1}{f_{p+1}} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1}, \tag{4.7}$$

and if $1 \leq j \leq p$*, then*

$$f_j T = 0. \tag{4.8}$$

Note that in case $f_i \in \tilde{\mathcal{O}}(Z)$, then the left hand sides of (4.7) and (4.8) are defined by (4.6).

Lemma 4.4. *Assume that* $f_1, f_2 \in \mathcal{O}(Z)$ *and that* $T \in \mathcal{PM}(Z)$ *is of bidegree* $(*, p)$*. If* $Z_{f_1} \cap Z_{f_2} \cap \operatorname{supp} T \subseteq V$*, for some analytic set* $V \subseteq Z$ *of codimension* $\geq p+1$ *in* Z*, then*

$$\frac{1}{f_1}\frac{1}{f_2}T = \frac{1}{f_2}\frac{1}{f_1}T. \tag{4.9}$$

If $Z_{f_1} \cap Z_{f_2} \cap \operatorname{supp} T \subseteq V$*, for some analytic set* V *of codimension* $\geq p+2$ *in* Z*, then*

$$\frac{1}{f_1}\bar\partial\frac{1}{f_2} \wedge T = \bar\partial\frac{1}{f_2} \wedge \frac{1}{f_1}T, \tag{4.10}$$

and if in addition $Z_{f_1} \cap Z_{f_2} \cap \operatorname{supp} \bar\partial T \subseteq V'$*, for some analytic set* V' *of codimension* $\geq p+3$*, then*

$$\bar\partial\frac{1}{f_1} \wedge \bar\partial\frac{1}{f_2} \wedge T = -\bar\partial\frac{1}{f_2} \wedge \bar\partial\frac{1}{f_1} \wedge T. \tag{4.11}$$

Proof. Outside of Z_{f_1}, we have by Proposition 3.1 that $(1/f_1)(1/f_2)T = (1/f_2)(1/f_1)T$, since both are just multiplication of $(1/f_2)T$ with the smooth function $(1/f_1)$, and similarly outside of Z_{f_2}. Thus $(1/f_1)(1/f_2)T - (1/f_2)(1/f_1)T$ is a pseudomeromorphic current on Z of bidegree $(*, p)$ with support on $Z_{f_1} \cap Z_{f_2} \cap V$, which has codimension $\geq p+1$, so (4.9) follows by Proposition 3.2. Similarly outside of Z_{f_1}, we get $(1/f_1)\bar\partial(1/f_2) \wedge T = \bar\partial(1/f_2) \wedge (1/f_1)T$, so $(1/f_1)\bar\partial(1/f_2) \wedge T - \bar\partial(1/f_2) \wedge (1/f_1)T$ is a pseudomeromorphic current on Z of bidegree $(*, p+1)$ and has support on $Z_{f_1} \cap Z_{f_2} \cap \operatorname{supp} T$, so (4.10) follows by Proposition 3.2. For (4.11), we get by Theorem 4.2 and (4.10) that

$$\begin{aligned}
&\bar\partial\frac{1}{f_1} \wedge \bar\partial\frac{1}{f_2} \wedge T = \bar\partial\left(\frac{1}{f_1}\bar\partial\frac{1}{f_2} \wedge T\right) + \frac{1}{f_1}\bar\partial\frac{1}{f_2} \wedge \bar\partial T \\
&= \bar\partial\left(\bar\partial\frac{1}{f_2} \wedge \frac{1}{f_1}T\right) + \frac{1}{f_1}\bar\partial\frac{1}{f_2} \wedge \bar\partial T \\
&= -\bar\partial\frac{1}{f_2} \wedge \bar\partial\frac{1}{f_1} \wedge T - \bar\partial\frac{1}{f_2} \wedge \frac{1}{f_1}\bar\partial T + \frac{1}{f_1}\bar\partial\frac{1}{f_2} \wedge \bar\partial T = -\bar\partial\frac{1}{f_2} \wedge \bar\partial\frac{1}{f_1} \wedge T
\end{aligned}$$

where the last equality holds because of (4.10) and the assumption of the support of $\bar\partial T$. □

Lemma 4.5. *Assume* $f, g \in \mathcal{O}(Z)$*, and* $f/g \in \mathcal{O}(Z)$*. If* $T \in \mathcal{PM}(Z)$ *has bidegree* $(*, p)$ *and* $Z_g \cap \operatorname{supp} T \subseteq V$*, for some analytic subset* V *of codimension* $\geq p+1$*, then*

$$f\left(\frac{1}{g}T\right) = \frac{f}{g}T.$$

Proof. Outside of Z_g, we can see $(1/g)T$ as multiplication by the smooth function $1/g$ by Proposition 3.1. Hence we have $f(1/g)T = (f/g)T$ since their difference is a pseudomeromorphic current with support on $Z_g \cap \operatorname{supp} T$, so it is 0 by Proposition 3.2. $\square$

Proof of Theorem 4.3. First we observe that it is enough to prove the theorem in case f_i are strongly holomorphic, since if $\pi : Z' \to Z$ is the normalization of Z, and $f' = \pi^* f$, then f' is a complete intersection, and if the theorem holds in Z', it holds in Z by taking push-forward of the corresponding currents. Hence, we can assume that $f_i \in \mathcal{O}(Z)$, and the commutativity properties will then follow from Lemma 4.4. For example, if we want to see that $1/f_{i+1}$ and $1/f_i$ commute, we can apply Lemma 4.4 with

$$T = \frac{1}{f_{i-1}} \cdots \frac{1}{f_{p+1}} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1},$$

and then multiply with $(1/f_m) \cdots (1/f_{i+2})$ from the left. In case some of the residue factors, say $f_{k+1}, \dots, f_p$, are to the left of the principal value factors, then $Z_{(f_1,\dots,f_k)}$ has codimension k in a neighborhood of $Z_f \supseteq \operatorname{supp} T$ and the result follows in the same way from Lemma 4.4. The other cases follow similarly from Lemma 4.4.

The equality (4.7) follows from Lemma 4.5 since Z_f has codimension p. By the first part, we can assume that $j = p$ in (4.8). Then

$$\begin{aligned} f_p \left(\bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} \right) &= \bar\partial \left(f_p \frac{1}{f_p} \wedge \bar\partial \frac{1}{f_{p-1}} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} \right) \\ &= \bar\partial \left(\bar\partial \frac{1}{f_{p-1}} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} \right) = 0 \end{aligned}$$

by (4.7), and Theorem 4.2. $\square$

5. Multiplication of currents with weakly holomorphic functions

Now, we will return to the issue of multiplication of currents with weakly holomorphic functions. Assume $g \in \tilde{\mathcal{O}}(Z)$, and $S \in \mathcal{PM}(Z)$. Since $S \in \mathcal{PM}(Z)$, we have $S = \sum (\pi_\alpha)_* \tau_\alpha$, where τ_α are elementary currents on the complex manifolds Z_α. Given such a decomposition, since any normal modification of Z factors through the normalization, we get a current S' in the normalization Z' of Z such that $\pi_* S' = S$ by taking the push-forward of τ_α to Z'. To define multiplication of the Coleff-Herrera product with the weakly holomorphic function g in (4.6), we defined it as the push-forward of $\pi^* g S'$. In general, the current S' will depend on the decomposition $S = \sum \pi_\alpha \tau_\alpha$. However, in (4.6) we had a canonical representative in the normalization, and hence the multiplication was well-defined. The following example however shows that it will in general depend not only on S, but also on the functions f defining S.

Example 3. Let $\pi : \mathbb{C}^n \to \mathbb{C}^{2n}$ be defined by

$$\pi(t_1, \cdots, t_n) = (t_1, \cdots, t_{n-1}, t_1^2 t_n, \cdots, t_{n-1}^2 t_n, t_n^2, t_n^5).$$

Then π is proper and injective, so $\pi(\mathbb{C}^n) = Z$ is an analytic variety of dimension n. Since $D\pi$ has full rank outside of $\{0\}$, $Z_{\text{sing}} \subseteq \{0\}$, and we will see below that actually $Z_{\text{sing}} = \{0\}$. Let

$$\tilde{S} = \bar\partial \frac{1}{t_1} \wedge \cdots \wedge \bar\partial \frac{1}{t_{n-1}} \wedge \bar\partial \frac{1}{t_n^3}$$

and $S = \pi_* \tilde{S}$. Then, since $d(t_n t_i^2) = t_i(2t_n dt_i + t_i dt_n)$ and $dt_n^5 = 5t_n^4 dt_n$, $dz_k \wedge S = 0$, where $k = n+i-1$, for $i = 1, \dots, n-1$ and $dz_{2n} \wedge S = 0$. Hence if $S.\xi \neq 0$, then ξ must be of the form $\xi = \xi_0 dz_1 \wedge \cdots \wedge dz_{n-1} \wedge dz_{2n-1}$. We have

$$S.\xi = \tilde{S}.\xi_0 dt_1 \wedge \cdots \wedge dt_{n-1} \wedge 2t_n dt_n =$$
$$2 \cdot (2\pi i)^n \left(\sum_{i=1}^{n-1} t_i^2 \frac{\partial}{\partial z_{n-1+i}} \xi_0 + 2t_n \frac{\partial}{\partial z_{2n-1}} \xi_0 + 5t_n^4 \frac{\partial}{\partial z_{2n}} \xi_0 \right) \Bigg|_{t=0} = 0,$$

and thus $S = 0$. However,

$$t_n \tilde{S}.\xi dt_1 \wedge \cdots \wedge dt_n^2 = 2(2\pi i)^n \xi(0)$$

so $\pi_*(t_n \tilde{S}) = \pi_*(\pi^* g \tilde{S}) \neq 0$, where $g \in \mathcal{O}_c(Z)$ is such that $\pi^* g = t_n$. Note that g is not strongly holomorphic at 0, and hence $Z_{\text{sing}} = \{0\}$.

From the above example we see that the definition of multiplication in (4.6) cannot be defined in terms of g and S only. In particular, if h is a tuple of holomorphic functions such that $P_g = \{h = 0\}$, then in general $g|h|^{2\lambda} S|_{\lambda=0}$ will not give the same definition of multiplication as the one in (4.6). However, there are cases where this definition works as the following proposition and its corollary show.

Proposition 5.1. *Let $\mu \in \mathcal{PM}(Z)$. Let $\phi \in \tilde{\mathcal{O}}(Z)$ and assume that h is a tuple of holomorphic functions such that $|h|^{2\lambda}\mu|_{\lambda=0} = \mu$ and $P_\phi = \{h = 0\}$, that is, $\mathbf{1}_{P_\phi}\mu = 0$. Then, the analytic continuation of $\phi|h|^{2\lambda}\mu$ to $\lambda = 0$ exists, and*

$$\phi\mu := \phi|h|^{2\lambda}\mu|_{\lambda=0}$$

is independent of the choice of such h.

Proof. Let $\mu = \sum(\pi_\alpha)_* \tau_\alpha$, where τ_α are elementary currents. By a resolution of singularities, we can assume that $\pi_\alpha^* h = h_0 h'$, where h_0 is a monomial and $|h'| \neq 0$. We can make a decomposition

$$\mu = \sum(\pi_\alpha)_* \tau'_\alpha + \sum(\pi_\alpha)_* \tau''_\alpha,$$

where the first sum consists of elementary currents such that τ'_α does not have any residue factors t_i with $\{t_i = 0\} \subseteq \{h_0 = 0\}$, and the

second sum consists of those that do. We can in fact assume that the second term is 0, because for $\operatorname{Re}\lambda \gg 0$, $\pi_\alpha^*|h|^{2\lambda}\tau_\alpha'' \equiv 0$, and hence

$$\mu = |h|^{2\lambda}\mu|_{\lambda=0} = \sum(\pi_\alpha)_*\tau_\alpha'.$$

Thus

$$\phi|h|^{2\lambda}\mu|_{\lambda=0} = \sum(\pi_\alpha)_*(\pi_\alpha^*(\phi|h|^{2\lambda})\tau_\alpha')|_{\lambda=0} = \sum(\pi_\alpha)_*(\pi_\alpha^*\tau_\alpha').$$

If g is another tuple of holomorphic functions such that $\{g=0\} = P_\phi$, then as above by a resolution of singularities we can assume that both $g = g_0 g'$ and $h = h_0 h'$, where g_0, h_0 are monomials and $|g'| \neq 0$, $|h'| \neq 0$. Since $\{h=0\} = \{g=0\}$, g_0 and h_0 are divisible by the same coordinates t_i. Hence as before, we can write $\mu = \sum(\pi_\alpha)_*\tau_\alpha'$, where τ_α' has no residue factors t_i from either g_0 or h_0. Then

$$\phi|g|^{2\lambda}\mu|_{\lambda=0} = \sum(\pi_\alpha)_*((\pi_\alpha^*\phi)\tau_\alpha') = \phi|h|^{2\lambda}\mu|_{\lambda=0}.$$

□

Corollary 5.2. *Assume $\mu \in \mathcal{PM}(Z)$ is of bidegree $(*,p)$ and $\phi \in \tilde{\mathcal{O}}(Z)$ is such that P_ϕ has codimension $\geq p+1$ in Z. Let h be any tuple of holomorphic functions such that $P_\phi = \{h=0\}$, and let μ' be any current in $\mathcal{PM}(Z')$ such that $\mu = \pi_*\mu'$, where $\pi : Z' \to Z$ is the normalization of Z. Then*

$$\phi\mu := \phi|h|^{2\lambda}\mu|_{\lambda=0} = \pi_*((\pi^*\phi)\mu').$$

Proof. By Proposition 3.2, we know that $\mu = |h|^{2\lambda}\mu|_{\lambda=0}$, since their difference must have support on $\{h=0\}$ which has codimension $\geq p+1$. Thus it follows from Proposition 5.1 that $\phi|h|^{2\lambda}\mu|_{\lambda=0}$ is independent of the choice of h. In addition, since outside $\{h=0\} = P_\phi$, we have that $\phi|h|^{2\lambda}\mu|_{\lambda=0}$ is just multiplication of μ with the smooth function ϕ and similarly for $\pi_*((\pi^*\phi)\mu')$. Thus it follows as above from Proposition 3.2 that they are equal. □

Note, in particular that if Z_{sing} has codimension $\geq p+1$, the condition of the codimension of P_g is automatically satisfied for any weakly holomorphic function $g \in \tilde{\mathcal{O}}(Z)$.

A related question is whether the Coleff-Herrera product could be defined as the analytic continuation of an integral on Z rather than Z'. We will see later that in the case of a complete intersection, this is in fact possible with the help of Bochner-Martinelli type residue currents, see Proposition 6.1 and Theorem 6.3. However, it is far from obvious that this definition coincides with the Coleff-Herrera current, and a more natural way to proceed would be to try to regularize in (4.3) by factors $\bar\partial|F_i|^{2\lambda_i}$ instead of $\bar\partial|f_i|^{2\lambda_i}$, where F_i is a tuple of strongly holomorphic functions such that $Z_{F_i} = P_{1/f_i}$. However, the analytic continuation to $\lambda = 0$ will in general not coincide with our definition, even if f defines a complete intersection, as the following example shows.

Example 4. Let $Z = \{z \in \mathbb{C}^3 \mid z_1^3 = z_2^2\} = V \times \mathbb{C}$, which has normalization $\pi(s,t) = (s^2, s^3, t)$, and let $\pi^* f_1 = s^2$ and $\pi^* f_2 = (1+s)t$. Then $Z_f = \{0\}$, so f is a complete intersection. Note that $\pi^*(1/f_2) = (1/t)(1 - s + O(s^2))$ for $|s| < 1$, and holomorphic functions in s at the origin correspond to strongly holomorphic functions on V at the origin precisely when the Taylor expansion at the origin contains no term s. Thus $P_{1/f_2} = \pi(\{s=0\} \cup \{s=1\} \cup \{t=0\})$, so if $\{F=0\} \supseteq P_{1/f_2}$, then $\{F = 0\} \supseteq Z_{f_1}$. Thus $(\bar\partial |F|^{2\lambda}/f_2) \wedge \bar\partial(1/f_1) = 0$ for $\operatorname{Re}\lambda \gg 0$. However, we have

$$\bar\partial\frac{1}{f_2} \wedge \bar\partial\frac{1}{f_1}.\varphi dz_1 \wedge dz_3 = \frac{1}{1+s}\bar\partial\frac{1}{t} \wedge \bar\partial\frac{1}{s^2}.\varphi(s^2, s^3, t) ds^2 \wedge dt = 4\pi i \varphi(0),$$

so $\bar\partial(1/f_1) \wedge \bar\partial(1/f_2)$ is non-zero.

6. Bochner-Martinelli type residue currents

We will show that we can define a Bochner-Martinelli type residue current associated with a tuple of weakly holomorphic functions, either by using a similar approach as for the Coleff-Herrera product with the help of the normalization, or by defining it intrinsically on Z by means of analytic continuation. In view of Example 4, it is not clear how to do this directly for the Coleff-Herrera product. In addition, we will show that for weakly holomorphic functions defining a complete intersection, the Coleff-Herrera product and the Bochner-Martinelli current coincide.

Let $f = (f_1, \dots, f_p)$ be weakly holomorphic. We will follow the approach by Andersson from [1], and make the identification $f = \sum f_i e_i^*$, where $(e_1, \cdots, e_p)$ is a frame for a trivial vector bundle E over Z. Since we will only use the case of trivial vector bundles, this identification merely serves as a notational convenience. Then, outside of the set where f is not strongly holomorphic, $\nabla_f = \delta_f - \bar\partial$ induces a complex on currents on Z with values in $\bigwedge E$, where δ_f is interior multiplication with f. To construct the Bochner-Martinelli current we define

$$\sigma = \sum \frac{\bar f_i e_i}{|f|^2} \quad \text{and} \quad u = \sum_{k=0}^{p-1} \sigma \wedge (\bar\partial\sigma)^k. \tag{6.1}$$

Note that outside of $Z_f \cup P_{f_1} \cup \dots \cup P_{f_p}$, both u and σ are smooth, and $\nabla_f u = 1$.

Proposition 6.1. *Assume that $f = (f_1, \cdots, f_p)$ is weakly holomorphic on Z. Let F be a strongly holomorphic function, or tuple of strongly holomorphic functions, such that $\{F = 0\} \supseteq Z_f \cup (\cup_{i=1}^p P_{f_i})$, and $\{F = 0\}$ does not contain any irreducible component of Z. Then the forms $|F|^{2\lambda} u$ and $\bar\partial |F|^{2\lambda} \wedge u$, originally defined for* $\operatorname{Re}\lambda \gg 0$*, have*

current-valued analytic continuations to $\operatorname{Re}\lambda > -\epsilon$ *for some* $\epsilon > 0$. *The currents*

$$U^f = |F|^{2\lambda}u|_{\lambda=0} \quad \textit{and} \quad R^f = \bar\partial|F|^{2\lambda} \wedge u|_{\lambda=0} \tag{6.2}$$

are independent of the choice of F, *and if* $\pi : Y \to Z$ *is a modification of* Z, *then* $U^f = \pi_* U^{\pi^* f}$ *and* $R^f = \pi_* R^{\pi^* f}$.

Proof. We first assume that Z is smooth. We can take $F = f$, and in that case, this is the existence part of Theorem 1.1 in [1], except for the fact that $U^f = \pi_* U^{\pi^* f}$ and $R^f = \pi_* R^{\pi^* f}$, which however easily follows by analytic continuation. To see that the definition of R^f is independent of the choice of F, we see from the proof of Theorem 1.1 in [1] that acting on a test-form φ, then $\bar\partial|F|^{2\lambda} \wedge u.\varphi$ becomes, with a suitable resolution of singularities $\pi : \tilde X \to X$, a finite sum of terms of the kind

$$\int \frac{\bar\partial|u\mu_1|^{2\lambda}}{\mu_2} \wedge \sigma' \wedge \pi^*\varphi, \tag{6.3}$$

where μ_1 and μ_2 are monomials such that $\{\mu_1 = 0\} \supseteq \{\mu_2 = 0\}$, u is non-zero and σ' is smooth. Thus, it is enough to observe that the value at $\lambda = 0$ of (6.3) is independent of μ_1 (which is the pull-back of F), as long as $\{\mu_1 = 0\} \supseteq \{\mu_2 = 0\}$. In the same way, one sees that the definition of U^f is independent of the choice of F. Now, if f is weakly holomorphic, and $\pi : \tilde Z \to Z$ is a resolution of singularities, from the smooth case we know that $\bar\partial|\pi^* F|^{2\lambda} \wedge \pi^* u$ has a current-valued analytic continuation to $\lambda = 0$ independent of the choice of $\pi^* F$. Hence, the weakly holomorphic case follows by taking push-forward, since $\bar\partial|F|^{2\lambda}u = \pi_*(\bar\partial|\pi^*F|^{2\lambda}\pi^* u)$ for $\operatorname{Re}\lambda \gg 0$. $\square$

The following properties of the Bochner-Martinelli current are well-known in the smooth case, see [21] and [1].

Proposition 6.2. *Let* $f = (f_1, \cdots, f_p)$ *be weakly holomorphic, and assume that* $p' = \operatorname{codim} Z_f$. *The current* R^f *has support on* $V = Z_f$, *and there is a decomposition* $R^f = \sum_{k=p'}^p R_k$, *where* $R_k \in \mathcal{PM}(Z)$ *is a* $(0,k)$*-current with values in* $\bigwedge^k E$. *In addition, if* f *is strongly holomorphic, then* $R^f = 1 - \nabla_f U^f$.

Proof. In case Z is a complex manifold, this is parts of Theorem 1.1 in [1], except for the fact that $R_k \in \mathcal{PM}(Z)$. However, the fact that R_k is pseudomeromorphic can, as was noted in [4], easily be seen from the proof of Theorem 1.1 in [1]. The proposition then follows in case of an analytic space, by taking push-forward from a resolution of singularities, except for the fact that $R^f = \sum_{k=p'}^p R_k$, where $p' = \operatorname{codim} Z_f$, since modifications does not in general preserve codimensions of subvarieties. However, we get that $R^f = \sum_{k=0}^p R_k$, where $R_k \in \mathcal{PM}(Z)$ is a $(0,k)$-current, and R_k has support on Z_f. Thus, by Proposition 3.2, $R_k = 0$ for $k < \operatorname{codim} Z_f = p'$. $\square$

Remark 4. If the mapping f is weakly holomorphic, as we saw in Example 3, we do not have a well-defined multiplication of weakly holomorphic functions with pseudomeromorphic currents on Z. However, since U^f is a principal value current, it is not hard to see that for any tuple h of holomorphic functions such that $h \not\equiv 0$, we have $U^f = |h|^{2\lambda} U^f|_{\lambda=0}$ (this follows as in the proof of Proposition 5.1, since if U^f is written as a sum of push-forwards of elementary currents, the elementary currents has no residue factors). Thus, by Proposition 5.1, we can define R^f by

$$R^f = 1 - \nabla_f(|F|^{2\lambda} U^f)|_{\lambda=0},$$

where F is a tuple of holomorphic functions such that $\{F = 0\} = P_f$, and hence R^f can be defined intrinsically on Z as the analytic continuation of a smooth form on Z.

Theorem 6.3. *If $f = (f_1, \dots, f_p)$ is weakly holomorphic forming a complete intersection and $R^f = \mu \wedge e$, where $e = e_1 \wedge \dots \wedge e_p$, then*

$$\mu = \mu^f := \bar\partial \frac{1}{f_p} \wedge \ldots \wedge \bar\partial \frac{1}{f_1}.$$

Proof. To begin with, we will assume that f is strongly holomorphic. The proof will follow the same idea as the proof in the smooth case in [2], Theorem 3.1. Let

$$V = \frac{1}{f_1} e_1 + \frac{1}{f_2} \bar\partial \frac{1}{f_1} \wedge e_1 \wedge e_2 + \dots + \frac{1}{f_p} \bar\partial \frac{1}{f_{p-1}} \wedge \dots \wedge \frac{1}{f_1} \wedge e_1 \wedge \dots \wedge e_p.$$

Then, by Proposition 4.3, V satisfies

$$\nabla_f V = 1 - \bar\partial \frac{1}{f_p} \wedge \dots \wedge \bar\partial \frac{1}{f_1} \wedge e.$$

Following the proof of Theorem 3.1 in [2], locally, assume $Z \subseteq \Omega \subseteq \mathbb{C}^n$, ω is an arbitrary neighborhood of Z in Ω and χ is a smooth function with support on ω which is $\equiv 1$ in a neighborhood of Z. Let $i : Z \to \Omega$ be the inclusion, and let $g = i^*\chi - i^*(\bar\partial\chi) \wedge u$. Then, since $\nabla_f u = 1$ outside of $\operatorname{supp} \bar\partial\chi$, $\nabla_f g = 0$, and hence

(6.4)
$$\nabla_f (g \wedge (U - V)) = g \wedge \nabla_f (U - V) = g_0(\mu^f - \mu) \wedge e = (\mu^f - \mu) \wedge e,$$

where $g_0 = \chi$ is the component of bidegree $(0, 0)$ in g, which is 1 in a neighborhood of $\operatorname{supp}(\mu^f - \mu)$. Since μ and μ^f are currents in $\mathcal{PM}(Z)$ of bidegree $(0, p)$, with support on $W = \{f = 0\}$, μ and μ^f has the standard extension property, SEP, (meaning that if h is a holomorphic function such that h is not identically 0 on any irreducible component of W, then $|h|^{2\lambda}\mu|_{\lambda=0} = \mu$) since if h does not vanish on any irreducible component of W, $\mu - |h|^{2\lambda}\mu|_{\lambda=0}$ has support on $W \cap \{h = 0\}$, which has codimension $\geq p + 1$, and by Proposition 3.2 it is 0. Also, μ and μ^f are $\bar\partial$-closed and are annihilated by $\bar I_W$, see Proposition 3.2, so $i_*\mu, i_*\mu^f \in CH_W$ (where CH_W denotes $\bar\partial$-closed $(0, \operatorname{codim} W)$-currents

with support on W satisfying the SEP). By Lemma 3.3 in [2], we know that a $\bar\partial$-closed current in CH_W cannot be equal to $\bar\partial\nu$, where ν can be chosen with support arbitrarily close to W, unless it is 0. Hence, by looking at the components of top degree in (6.4), we have $i_*(\mu - \mu^f) = 0$, so $\mu = \mu^f$. Now, if f_i are weakly holomorphic, then the current R^f will be the push-forward of the corresponding current $R^{\pi^* f}$, where $\pi : Z' \to Z$ is the normalization of Z, and the same holds for the Coleff-Herrera product μ^f. Hence, equality holds in the normalization, and taking push-forward we get equality in the general case. $\square$

7. The transformation law

With the Bochner-Martinelli type currents developed in the previous section, we will now prove the transformation law for Coleff-Herrera products of weakly holomorphic functions. The transformation law is proven by showing that the corresponding Bochner-Martinelli currents coincide, and then we use the equality of the Bochner-Martinelli current and the Coleff-Herrera product. In [13] Denkowski proved the transformation law for c-holomorphic functions based on a more direct approach.

Theorem 7.1. *Assume that $f = (f_1, \cdots, f_p)$ and $g = (g_1, \cdots, g_p)$ are weakly holomorphic, defining complete intersections, and that there exists a matrix A of weakly holomorphic functions such that $g = Af$. Then*

$$\bar\partial\frac{1}{f_p} \wedge \cdots \wedge \bar\partial\frac{1}{f_1} = (\det A)\bar\partial\frac{1}{g_p} \wedge \cdots \wedge \bar\partial\frac{1}{g_1}.$$

If A is invertible, one can prove the transformation law with the help of Theorem 6.3 together with the fact that the Bochner-Martinelli current is independent of the metric chosen to define σ^f (here, in (6.1), σ^f is defined with respect to the trivial metric on E), see [1]. We will see that we can use a similar idea even in the case that A is not invertible.

To begin with, we assume that f, g and A consist of strongly holomorphic functions. As in the previous section, we will identify f and g with sections of vector bundles, however we will here identify them with sections of two different vector bundles. Let E and E' be trivial holomorphic vector bundles over Z with frames e and e', and make the identifications $f = \sum f_i e_i^*$, $g = \sum g_i e_i'^*$ and $A \in \operatorname{Hom}(E', E)$ such that $g = fA$.

Lemma 7.2. *Let $\bigwedge A : \bigwedge E' \to \bigwedge E$ denote the linear extension of the mapping $(\bigwedge A)(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$. Then $\delta_f(\bigwedge A) = (\bigwedge A)\delta_g$.*

Proof. Note first that $\delta_f Ae'_j = g_j = \delta_g e'_j$. Hence, we have

$$\begin{aligned}
&\delta_f(\bigwedge A)(e'_{i_1}\wedge\cdots\wedge e'_{i_k}) = \delta_f(Ae'_{i_1}\wedge\cdots\wedge Ae'_{i_k})\\
&=\sum(-1)^{j-1}Ae'_{i_1}\wedge\cdots\wedge\delta_f(Ae'_{i_j})\wedge\cdots\wedge Ae'_{i_k}\\
&=\sum(-1)^{j-1}(\bigwedge A)(e'_{i_1}\wedge\cdots\wedge\delta_g e'_j\wedge\cdots\wedge e'_{i_k}) = (\bigwedge A)\delta_g(e'_{i_1}\wedge\cdots\wedge e'_{i_k}).
\end{aligned}$$

□

To relate the currents μ^f and μ^g, we will first derive a relation between the currents U^f and U^g as defined by (6.2).

Lemma 7.3. *If f and g are strongly holomorphic and defining complete intersections, then there exists a current R_1 such that $U^f - (\bigwedge A)U^g = \nabla_f R_1$.*

Proof. Let σ, u, σ' and u' be the forms defined by (6.1) corresponding to f and g. Since A is holomorphic, $(\bigwedge A)\bar\partial\sigma' = \bar\partial(A\sigma')$ outside of $\{g=0\}$, and hence if we let $u'_A = \sum(A\sigma')\wedge(\bar\partial A\sigma')^{k-1}$, then $\nabla_f u'_A = 1$ outside of $\{g=0\}$ by Lemma 7.2. Thus, if $\operatorname{Re}\lambda \gg 0$,

$$\nabla_f(|g|^{2\lambda}u'_A\wedge u) = |g|^{2\lambda}u - |g|^{2\lambda}u'_A - \bar\partial|g|^{2\lambda}\wedge u'_A\wedge u. \tag{7.1}$$

We want to see that all the terms in (7.1) have current-valued analytic continuations to $\lambda = 0$. First, we note that since $\{g=0\}\supseteq\{f=0\}$, $|g|^{2\lambda}u|_{\lambda=0} = U^f$ by Proposition 6.1, and since $u'_A = (\bigwedge A)u'$ we get that $|g|^{2\lambda}u'_A|_{\lambda=0} = (\bigwedge A)U^g$. Thus it remains to see that the left hand side of (7.1) has an analytic continuation to $\lambda=0$, and that the analytic continuation of the last term vanishes at $\lambda=0$. To see that those terms have analytic continuations to $\lambda=0$ is similar to showing the existence of the Bochner-Martinelli currents U^f and R^f. If we recall briefly the proof of the existence of U^f and R^f in [1], the key step was that $\sigma\wedge(\bar\partial\sigma)^{k-1}$ is homogeneous with respect to f in the sense that if $f = f_0 f'$, then $\sigma\wedge(\bar\partial\sigma)^{k-1} = (1/f_0^k)\sigma_0\wedge(\bar\partial\sigma_0)^{k-1}$, where σ_0 is smooth if $|f'|\neq 0$. By blowing up along the ideals $(f_1,\dots,f_p)$ and $(g_1,\dots,g_p)$ followed by a resolution of singularities, see [5], we can assume that locally $\pi^*f = f_0 h$ and $\pi^*g = g_0 g'$, where $h\neq 0$, $g'\neq 0$, and by a further resolution of singularities, we can assume that locally f_0, g_0 are monomials. Since $\{g=0\}\supseteq\{f=0\}$, we get that $\{g_0=0\}\supseteq\{f_0=0\}$. Thus, by the homogeneity of $\sigma'\wedge(\bar\partial\sigma')^{k-1}$ and $\sigma\wedge(\bar\partial\sigma)^{l-1}$ with respect to f and g, we get, since $u'_A = (\bigwedge A)u'$, that $|g|^{2\lambda}u'_A\wedge u$ and $\bar\partial|g|^{2\lambda}\wedge u'_A\wedge u$ acting on a test form φ becomes finite sums of the form

$$\int\frac{|v|^{2\lambda}|g_0|^{2\lambda}}{(g_0)^k f_0^l}\xi_{k,l}\wedge\pi^*\varphi \quad\text{and}\quad \int\frac{\bar\partial(|v|^{2\lambda}|g_0|^{2\lambda})}{(g_0)^k f_0^l}\wedge\xi_{k,l}\wedge\pi^*\varphi,$$

where $\xi_{k,l}$ are smooth $(0,k+l-2)$-forms. Thus both have analytic continuations to $\lambda=0$, and $R_2 := \bar\partial|g|^{2\lambda}\wedge u'_A\wedge u|_{\lambda=0}$ has support on $\{g=0\}$. Since $R_2\in\mathcal{PM}(Z)$ and consists of terms of bidegree

$(0, k+l-1)$, where $k+l \leq p$, with support on $\{g=0\}$ which has codimension p, we get that $R_2 = 0$ by Proposition 3.2. Thus, if we let $R_1 := |g|^{2\lambda} u'_A \wedge u|_{\lambda=0}$, we get that $\nabla_f R_1 = U^f - (\bigwedge A)U^g$. $\square$

Now we are ready to prove the transformation law.

Proof of Theorem 7.1. Assume first that f, g and A are strongly holomorphic, and make the identifications as above. Since $(\bigwedge A)R^g = (\bigwedge A)(1 - \nabla_g U^g) = 1 - \nabla_f (\bigwedge A)U^g$ by Lemma 7.2, we get from Lemma 7.3 that

$$(\bigwedge A)R^g - R^f = \nabla_f \left((\bigwedge A)U^g - U^f \right) = \nabla_f^2 R_1 = 0,$$

so

$$(\bigwedge A)R^g = R^f.$$

Thus, we get by Theorem 6.3 that

$$(\bigwedge A) \left(\mu^g \wedge e'_1 \wedge \cdots \wedge e'_p \right) = \mu^f \wedge e_1 \wedge \cdots \wedge e_p,$$

and since the left hand side is equal to

$$(\det A) \mu^g \wedge e_1 \wedge \cdots \wedge e_p,$$

the transformation law follows. Now, if f, g and A are weakly holomorphic, the transformation law follows since equality must hold in the normalization because the pullback of f and g define complete intersections in the normalization, and hence equality must hold also in Z by taking push-forward. $\square$

8. The Poincaré-Lelong formula

Let $f_1, \cdots, f_p$ be strongly holomorphic functions forming a complete intersection. The Poincaré-Lelong formula says that

$$\frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} \wedge df_1 \wedge \cdots \wedge df_p = [Z_f] = \sum \alpha_i [V_i], \tag{8.1}$$

where V_i are the irreducible components of Z_f and $[Z_f]$ is the integration current on Z_f with multiplicities. In case $p = \dim Z$ the multiplicities α_i are given as the number of elements of a generic fiber of f near a given point, and in case $p < \dim Z$ the multiplicity is given as the intersection multiplicity of Z_f with L, where L is a plane of dimension $\dim Z - p$ transversal to Z_f. For a thorough discussion of the multiplicities see [11], and for a proof of the Poincaré-Lelong formula see Section 3.6 in [12].

Now, if f_i are weakly holomorphic functions defining a complete intersection, we can give a relatively short proof that a similar formula holds in Z. In the strongly holomorphic case, assuming $Z \subseteq \Omega \subseteq \mathbb{C}^n$, $i_*[Z_f]$ can be seen either as the intersection of the holomorphic chains

Z_{F_i} with Z, where F_i are some holomorphic extensions of f_i to Ω, or as a product of closed positive currents, see [11], that is

$$i_*[Z_f] = [Z_{F_1} \cdot \dots \cdot Z_{F_p} \cdot Z] = [Z_{F_1}] \wedge \dots \wedge [Z_{F_p}] \wedge [Z].$$

However, these types of products are in general only defined in case $Z_{F_1} \cap \dots \cap Z_{F_p} \cap Z$ has codimension equal to $\operatorname{codim} Z + \sum \operatorname{codim} Z_{F_i}$. Since we saw in Example 1 that zero sets of weakly holomorphic functions are in general not zero sets of strongly holomorphic functions, we cannot expect to have a similar interpretation for weakly holomorphic functions.

Let $\pi : Z' \to Z$ be the normalization of Z. Since $f' = \pi^* f$ forms a complete intersection, (8.1) holds for f' in the normalization. Note that, since π is the normalization of Z, π is a finite proper holomorphic map and $\pi(V_i) = W_i$ are irreducible in Z. If $f : V \to W$ is a branched holomorphic cover with exceptional set E, we say that f is a **-covering* if $W \setminus E$ is a connected manifold. In particular, this means that the sheet-number of f is constant outside the exceptional set. By the Andreotti-Stoll theorem, see [17], if $f : V \to W$ is a finite proper holomorphic map, V has constant dimension and W is irreducible, then f is a *-covering. Hence, if $V \subset Z'$ is an irreducible component of $Z_{f'}$ and we consider $\pi|_V : V \to W$, where $W = \pi(V)$, it is a finite proper holomorphic map satisfying the conditions required for the Andreotti-Stoll theorem. Hence, there exists an integer k such that $\pi|_V$ is a k-sheeted finite branched holomorphic covering. Thus $\pi_* \alpha[V] = k\alpha[W]$. For $f = (f_1, \cdots, f_p)$ a weakly holomorphic mapping forming a complete intersection, we define the left-hand side of (8.1) as the push-forward of the corresponding current in the normalization. Thus, since we have by (8.1) that

$$\frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_p} \wedge \dots \wedge \bar\partial \frac{1}{f_1} \wedge df_1 \wedge \dots \wedge df_p = \pi_*[Z_{f'}],$$

we have proved the following.

Theorem 8.1. *Let $f = (f_1, \cdots, f_p)$ be a weakly holomorphic mapping forming a complete intersection. Then*

$$\frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_p} \wedge \dots \wedge \bar\partial \frac{1}{f_1} \wedge df_1 \wedge \dots \wedge df_p = \sum \beta_i [W_i] \tag{8.2}$$

where $\beta_i \in \mathbb{N}$ and W_i are the irreducible components of $W = Z_f$. More explicitly, if $[Z_{f'}] = \sum \alpha_i [V_i]$ and say $V_{i_1}, \cdots, V_{i_k}$ are the sets V_j such that $\pi(V_j) = W_i$, then $\beta_i = \sum k_{i_j} \alpha_{i_j}$, where k_j is the number of elements in a generic fiber of $\pi|_{V_j}$.

Remark 5. In [13] Denkowski proves the Poincaré-Lelong formula for $f = (f_1, \dots, f_p) \in \mathcal{O}_c^{\oplus p}(Z)$ (based on his construction on Γ_f, however as for the Coleff-Herrera product in Proposition 4.1 our definition coincides with his), and in that case it gives a different interpretation of

the multiplicities as the intersection cycle

$$\frac{1}{(2\pi i)^p}\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge df_1\wedge\cdots\wedge df_p=\pi_*([\Gamma_f]\cdot[Z\times\{0\}]),$$

where $\pi : Z\times\mathbb{C}^p\to Z$ is the projection.

Note that if f is weakly holomorphic, since f is in general not smooth on Z_{sing}, df is not in general defined on all Z (although its pullback to the normalization has a smooth extension to all of Z') so, as for multiplication with weakly holomorphic functions in Example 3, it might for example happen that $\bar\partial(1/f)=0$ while $\bar\partial(1/f)\wedge df\neq 0$. For example, if $Z=\{z^3=w^2\}$, $\pi(t)=(t^2,t^3)$ and $f=w/z\in\tilde{\mathcal{O}}(Z)$, that is $\pi^*f=t$, then $\bar\partial(1/f)=0$ while $\bar\partial(1/f)\wedge df=2\pi i[0]$, as expected, since $Z_f=\{0\}$.

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